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p-ADIC MODULAR FORMS

Modular forms are classically defined as certain functions on the complex upper half-plane $\mathcal{H} = \{\mathrm{Im}\ \tau > 0\}$ satisfying certain strong transformation properties, induced by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{C})$, which preserves the upper half-plane. In particular, they satisfy invariance under $\tau \mapsto \tau + 1$, and hence admit q-expansions

$$f(\tau) = \sum a_n q^n$$

where $q = \exp(2\pi i\tau)$. While this definition is purely complex-analytic, modular forms have strong arithmetic properties; in particular, the a_n can all simultaneously be in $\mathbb{Q}$. In particular, base changing to the p-adic numbers $\mathbb{Q}_p$, it makes sense to study modular forms p-adically.

This is most conceptually understood in terms of the interpretation of modular forms as sections of line bundles on the moduli stack of elliptic curves. The moduli stack of elliptic curves is defined over $\mathbb{Z}$, and (the analytification of) its base change to $\mathbb{C}$ is given by $\mathcal{H}/\mathrm{SL}_2(\mathbb{Z})$. The theory of p-adic modular forms is then closely linked to the moduli stack of elliptic curves over $\mathbb{Z}_p$ and $\mathbb{Q}_p$, especially regarded as a p-adic analytic space.

The story of p-adic modular forms is by now quite classical, going back to works of Serre, Katz, and others. Serre realized that modular forms can vary naturally in interesting p-adic families; namely, there are natural families of classical modular forms f_k of weight k for which the coefficients $a_n = a_{n,k}$ of the q-expansion of f_k can be interpolated p-adically (in k). Specializing to p-adic values of k, one has genuinely new p-adic modular forms. Katz realized that these could be understood in terms of sections of line bundles on the Igusa curve, a certain cover of the ordinary locus inside the moduli stack of elliptic curves over $\mathrm{Spf}\ \mathbb{Z}_p$.

Often, natural examples of such p-adic modular forms converge even slightly outside the ordinary locus, leading to so-called overconvergent p-adic modular forms. There has been a tremendous amount of work on these various topics, and the relations to the (p-adic) Hodge theory of modular curves. The goal of this seminar is to study some recent papers, revolving around the infinite-level modular curve, the Hodge-Tate period map, and perfect Igusa varieties.

TALKS

Talk 1: Modular curves and modular forms

Recall the basics on modular curves (with "enough" level structure), as complex analytic spaces and then as schemes. Recall the classical points of view on modular forms: as holomorphic functions, as section of line bundles, and as functions on test objects following Katz. Recall Hecke operators, as cohomological correspondences and as explicit operations on q-expansions. References: [7, 8, 11]

Talk 2: p-adic geometry of modular curves

Recall the definition of the ordinary locus and the "standard" strict neighborhoods of its rigid generic fiber. Define the notion of canonical subgroups. Sketch the proof that a canonical subgroup of order p^n exists on an explicit strict neighborhood of the ordinary locus. Recall the classical Igusa tower over the ordinary locus, and perfect Igusa varieties as in [2] (following [5]). References: [7], [5], [13]

Talk 3: p-adic modular forms and q-expansions

Recall Katz's definition of p-adic modular forms as functions on the Igusa tower, and explain the relation with classical modular forms and q-expansions. Explain the $\widehat{\mathbf{G}_m}$-action on p-adic modular forms defined in [5], and what it does on q-expansions. References: [7], [5]

Talk 4: Hida theory

Recall Hida's definition of the ordinary projector and the space of p-ordinary p-adic modular forms. Sketch the proofs that p-ordinary cusp forms of integral weights $k \geq 2$ are classical, and that the space of p-ordinary forms is finite projective over Λ, following the "H^0 part" of [1, Section 4].

Talk 5: Higher Hida theory

Explain the H^1 part of [1, Section 4].

Talk 6: The infinite-level modular curve and the Hodge–Tate period map

Recall the infinite-level perfectoid modular curve $\mathcal{X}_\infty$ and its Hodge–Tate period map. Explain how the Hodge–Tate period map gives explicit strict neighborhoods of the ordinary locus, and compare them to the standard ones, following [3]. Present the definition of overconvergent modular forms as functions on an affinoid piece of $\mathcal{X}_\infty$, following [3] (and ignoring cusps as necessary). Explain the analogies in [3, Table 1.1].

References: [13], [3], [9]

Talk 7: Overconvergent modular forms

Compare the definition of overconvergent modular forms given in [3] with Pilloni's definition [10]. Sketch Heuer's simplified proof of descent of the line bundle from infinite level. Explain why the U_p-operator is compact.

References: [3], [10], [1, Section 5.5]

Talk 8: Completed cohomology

Recall the definition of completed cohomology, and give some motivation in terms of the p-adic Langlands program. Explain primitive comparison isomorphisms, and the description of completed cohomology as the cohomology on $\mathbb{P}^1$ of $\pi_* \mathcal{O}_{K^p}$.

References: [4, 12, 14]

Talk 9: Completed cohomology and overconvergent modular forms

Explain the main results of the paper [6].

Talks 10-11: Locally analytic vectors in completed cohomology

Say something about the main results of the paper [14]. In particular, describe the geometric Sen theory developed there, its relation with p-adic Simpson, and how it applies to give a "Cauchy–Riemann type" differential equation for the sheaf of locally analytic sections.

References

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