ARGOS - Arithmetische Geometrie Oberseminar

Arithmetic Bigness and uniform Bogomolov

Programmvorschlag: A. Mihatsch, P. Scholze

Bonn, Winter Term 2021/22

The seminar's aim is to understand the recent bigness result of Yuan and its application to the Bogomolov conjecture. First recall the latter:

Theorem (Bogomolov Conjecture; Ullmo and Zhang 1998)**.** *Let K be a number field and C/K a curve of genus $g \geq 2$. Then there are $c_1, c_2 > 0$ such that*

$$\#\{x \in C(\overline{K}) \mid h_{\mathrm{NT}}(x) < c_1\} \leq c_2.$$

Here, $h_{\mathrm{NT}}\colon C(\overline{K}) \to \mathbb{R}_{\geq 0}$ is the Néron–Tate height. It essentially measures the number of digits required to write down the $\overline{K}$-point in question, with respect to a specific choice of projective embedding of C. The theorem thus says that there are only finitely many small points on C.

S.-W. Zhang has put the Bogomolov Conjecture into the realm of Arakelov theory. He endows the canonical bundle $\omega = \omega_{C/K}$ with suitable metrics g_v at all places v of K. The metrics at archimedean v are the famous Arakelov metrics on the Riemann surface $C(\mathbb{C}_v)$. At non-archimedean v however, the metric lives on the Berkovich space of C at v and is related to the reduction of C at v. Writing $\bar{\omega}_a$ for this metrized version of ω, the Bogomolov Conjecture is essentially equivalent to positivity of its self-intersection,

$$(\bar{\omega}_a, \bar{\omega}_a) > 0.$$

This is the "arithmetic bigness" of the seminar title.

The uniform Bogomolov Conjecture states that the above constants c_1 and c_2 may be chosen independently of the curve, only depending on g. Yuan's proof is by considering the relative dualizing sheaf ω_{C/M_g} of the universal curve C over the moduli space of smooth genus g curves M_g. He shows that the above metrics fit into a continuous family and that the so-defined metrized line bundle $\bar{\omega}_{C/M_g,a}$ has a bigness property over M_g, or after pullback along any generically finite map $S \to \mathcal{M}_g$. (Before Yuan's work, Dimitrov–Gao–Habegger and Kühne have also obtained results on the uniform Bogomolov conjecture. Moreover, by known bounds on the number of large points, the uniform Bogomolov conjecture implies a uniform version of the Mordell conjecture, with the bound on the number of points depending only on g and the rank of the Jacobian.)

The heart of Yuan's proof is a careful study of the asymptotics of the involved metrics towards the boundary $\overline{M}_g \setminus M_g$ of M_g. Loosely speaking, the key is to bound the pace of degeneration of C when approaching the boundary. A key part of the proof goes back to Cinkir; it has a combinatorial flavour because numerical invariants are derived from the reduction graph of the curve.

Talks

The program mostly follows Yuan [4]. References point to this article if not stated otherwise.

Talk 1: The Bogomolov Conjecture

Introduce heights as in [3, Chapter VI]. Recall the definition of the Néron–Tate height for abelian varieties and its properties. Formulate the Bogomolov and Manin–Mumford conjectures. References: [3, Chapter VI] and [7, §1–§2].

Talk 2: Moduli of Curves

Introduce the moduli space M_g of curves of genus $g \geq 2$, following the survey and references in §3.1. Explain its compactification by the moduli space of stable curves and the description of the boundary divisor. Define the Hodge bundle and discuss Noether's formula. Sketch the bigness of the Hodge bundle as on p. 46. References: [2] as well as [4, §3.1 and p. 46].

Talk 3: Adelic line bundles

Explain the definition of adelic line bundles and their intersection theory, following the survey given in §2.1. In particular, introduce the Deligne pairing of adelic line bundles. References: [4, §2.1] and [5].

Talk 4: Admissible canonical bundle I

Present the definition and construction of the admissible metrics as in Thm. A.1 of the appendix. Stick to the non-archimedean theory in §A.2–§A.6; the archimedean theory in §A.1 will be part of a later talk. Reference: [4, Appendix].

Talk 5: Admissible canonical bundle II

Prove the existence and properties of the admissible metrics for families of curves as in Thm. 2.3. Reference: [4, §2.2].

Talk 6: Intersection and pull-back

The goal of this talk is to present some properties of admissible adelic line bundles as in §2.3. These are an adjunction formula, a Hodge index theorem and relations between the admissibly metrized dualizing sheaf $\bar{\omega}_{X/S,a}$ and Θ-divisors. The key takeaway is that bigness of $\bar{\omega}_{X/S,a}$ implies bigness of various other adelic line bundles. Reference: [4, §2.3].

Talk 7: The relative dualizing sheaf

Explain the comparison of the admissible adelic line bundle $\bar{\omega}_{X/S,a}$ and the relative dualizing sheaf $\omega_{\overline{X}/\overline{S}}$ of a given stable compactification $\overline{X}/\overline{S}$. Reference: [4, §3 up to Lem. 3.4]

Talk 8: The φ-invariant

Say something about Zhang's φ-invariant. It is a measure for the degree of degeneracy of a curve at some place. Then prove Thm. 3.5 which relates three terms: The self-pairing of the admissible canonical bundle, the φ-invariant and the Θ-divisor. References: [4, §3.3] and [6].

Talk 9: Cinkir's bound

Explain how to bound the φ-invariant in terms of the boundary divisor. This is Thm. 3.7 in [4] and due to Cinkir. References: [1] and [4, §3.3].

Talk 10: Bigness of the canonical bundle

Quickly combine the results of the previous two talks to obtain the "bigness in the geometric case" (Thm. 3.8). Then introduce the admissible metrics at archimedean places from §A.1. Explain the proof of the bigness in the arithmetic case, Thm. 3.9. The additional ingredient here is a bound on the φ-invariant also at archimedean places. Reference: [4, §3].

Talk 11: Potentially big line bundles

Explain the abstract results on potentially big line bundles from §4.1. In particular, explain the relation of bigness with bounds on the number of small points. Present the examples of potentially big line bundles from §4.2. Reference: [4, §4.1 and §4.2].

Talk 12: Uniform Bogomolov

Finish the proof of the uniform Bogomolov conjecture along the lines of §4.3 and §4.4. Reference: [4, §4].

References

[1] Z. Cinkir, *Zhang's conjecture and the effective Bogomolov conjecture over function fields*, Invent. Math. **183** (2011), no. 3, 517–562.

[2] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. Inst. Hautes Études Sci. **36** (1969), no. 1, 75–109.

[3] G. Cornell and J. H. Silverman (eds.), *Arithmetic Geometry*, Springer New York, 1986.

[4] X. Yuan, *Arithmetic bigness and a uniform Bogomolov-type result*, Preprint arXiv:2108.05625.

[5] X. Yuan and S.-W. Zhang, *Adelic line bundles over quasi-projective varieties*, Preprint arXiv:2105.13587.

[6] S.-W. Zhang, *Gross–Schoen Cycles and Dualising Sheaves*, Preprint arXiv:0812.0371 (2008).

[7] ———, *Small points and Arakelov theory*, Doc. Math. **Extra Volume ICM 1998** (1998), no. 2, 217–225.