Summer term 2018
Real and harmonic analysis
Exercise sheet 4

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Submit the solutions in groups of two at the lecture on Tuesday, 2018-05-08

Exercise 1 (Fefferman–Stein inequality)**.** Define the maximal function of a non-negative and non-zero regular Borel measure μ as

$$M\mu(x) = \sup_{B \ni x} \frac{\mu(B)}{|B|}$$

where the supremum is taken over all Euclidean balls $B \subset \mathbb{R}^d$.

(a) Show that $M\mu$ is measurable and $L^1_{loc}(M\mu) \subset L^1_{loc}(\mathrm{d}x)$ where $\mathrm{d}x$ denotes the usual Lebesgue measure.

(b) Let $\nu \neq 0$ be another non-negative Borel measure. Show that

$$\mu(\{x \in \mathbb{R}^d : M\nu(x) > \lambda\}) \leq \frac{C_d}{\lambda} \int_{\mathbb{R}^d} M\mu \, \mathrm{d}\nu$$

for all $\lambda > 0$.

Exercise 2 (Vector valued extension)**.** (a) Let n be a positive integer, $\lambda \in \mathbb{R}^n$ and $p \in (0, \infty)$. Show that

$$\left(\int_{\mathbb{R}^n} |\lambda \cdot x|^p e^{-\pi |x|^2} \mathrm{d}x \right)^{1/p} = C_p |\lambda|.$$

Note that C_p is independent of n.

(b) Let (X, μ) be a measure space. Suppose that $T : L^p(X) \to L^p(X)$ is a bounded linear operator. Let $f_j \in L^p(X)$ for all $j \in \mathbb{Z}$. Show that

$$\|(\sum_j |Tf_j|^2)^{1/2}\|_p \leq \|T\| \cdot \|(\sum_j |f_j|^2)^{1/2}\|_p.$$