46 Vortrag Bonn

46.1 The ρ-invariant

46.1.1 The regulator

Let $\mathbf{K}(\mathbb{C})$ be the algebraic K-theory spectrum of $\mathbb{C}$ and $\mathbf{KU}\mathbb{C}/\mathbb{Z}$ be the complex K-theory spectrum with coefficients in $\mathbb{C}/\mathbb{Z}$. There is map

$$\mathtt{reg}_{\mathbb{C}} : \mathbf{K}(\mathbb{C})[1..\infty] \to \Sigma^{-1}\mathbf{KU}\mathbb{C}/\mathbb{Z} \tag{22}$$

of spectra such that

$$\mathtt{reg}_{\mathbb{C},2m-1} : \pi_{2m-1}(\mathbf{K}(\mathbb{C})) \to \pi_{2m}(\mathbf{KU}\mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}$$

detects the torsion subgroup $\pi_{2m-1}(\mathbf{K}(\mathbb{C}))_{tors} \cong \mathbb{Q}/\mathbb{Z}$ for all $m \in \mathbb{N}$, $m \geq 1$. Furthermore,

$$\mathtt{reg}_{\mathbb{C},1} : \mathbb{C}^* \cong \pi_1(\mathbf{K}(\mathbb{C})) \to \pi_2(\mathbf{KU}\mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}$$

is the isomorphism given by $-\frac{1}{2\pi i}\log$.

46.1.2 Flat bundles and algebraic K-theory classes

Let B be a closed n-dimensional manifold with a stable framing of its tangent bundle

$$s : TB \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}}^{k+n} \ .$$

By the Thom-Pontrjagin construction it represents a homotopy class $[B, s] \in \pi_n(\mathbf{S})$ of the sphere spectrum.

We consider a complex ℓ-dimensional vector bundle $V \to B$ with a flat connection ∇. It can be considered as a bundle with the structure group $GL(\ell, \mathbb{C}^\delta)$. From its classifying map we get a morphism of spaces

$$v : B \to BGL(\ell, \mathbb{C}^\delta) \to BGL(\mathbb{C}^\delta)^+ \simeq \Omega^\infty\mathbf{K}(\mathbb{C}) \ .$$

Again by the Thom-Pontrjagin construction it represents a class

$$[B, s, v] \in \pi_n(\mathbf{S} \wedge \Omega^\infty\mathbf{K}(\mathbb{C})) \ ,$$

and we let

$$c(B, \nabla, s) \in \pi_n(\mathbf{K}(\mathbb{C}))$$

be the algebraic K-theory class defined by its stabilization. One can show that every element in $\pi_n(\mathbf{K}(\mathbb{C}))$ can be presented in this way (Jones-Westbury 95).

46.1.3 The analytic side

We choose a hermitean metric h on V and define the corresponding unitarization ∇^u of ∇. The triple $\mathbf{V} = (V, h, \nabla^u)$ is called a geometric bundle.

The manifold B has a spin structure determined by the framing. We choose a Riemannian metric g^B and consider the twisted Dirac operator $\not{D}_B \otimes \mathbf{V}$. It is a selfadjoint elliptic differential operator of first order acting on sections of $S(B) \otimes V \to B$, where $S(B) \to B$ is the spinor bundle. It has a discrete spectrum $\sigma(\not{D}_B \otimes \mathbf{V}) \subset \mathbb{R}$ consisting of eigenvalues of finite multiplicity. The eta function is defined

$$\eta_{\not{D}_B \otimes \mathbf{V}}(s) := \sum_{\lambda \in \sigma(\not{D}_B \otimes \mathbf{V}) \setminus \{0\}} \mathtt{sign}(\lambda) \mathrm{mult}(\lambda) |\lambda|^{-s} .$$

The series is convergent and holomorphic for $\mathtt{Re}(s) > n$ and has a meromorphic extension to all of $\mathbb{C}$ which is regular at 0. The η-invariant of $\not{D}_B \otimes \mathbf{V}$ is defined by

$$\eta(\not{D}_B \otimes \mathbf{V}) = \eta_{\not{D}_B \otimes \mathbf{V}}(0) .$$

It has been introduced by Atiyah-Patodi-Singer as a boundary correction term of an index theorem for Dirac operators on a manifold with boundary. We further define the reduced η-invariant

$$\xi(\not{D}_B \otimes \mathbf{V}) := [\frac{\eta(\not{D}_B \otimes \mathbf{V}) - \dim \mathtt{ker}(\not{D}_B \otimes \mathbf{V})}{2}]_{\mathbb{C}/\mathbb{Z}} .$$

Definition 46.1. *We define*

$$\rho(B, \nabla, s) := \xi(\not{D}_B \otimes \mathbf{V}) + \left[\int \hat{\mathbf{A}}(\nabla^{LC}) \wedge \widetilde{\mathbf{ch}}(\nabla, \nabla^u) - \tilde{\hat{\mathbf{A}}}(\nabla^{LC}, \nabla^s) \wedge \mathbf{ch}(\nabla)\right]_{\mathbb{C}/\mathbb{Z}} .$$

This quantity is independent of the auxiliary choices of the Riemannian metric on B and the unitary connection on V.

As an example we consider $B = S^1 = \mathbb{R}/\mathbb{Z}$ with the induced Riemannian metric, standard framing s and the one-dimensional unitary flat bundle $\mathbf{V} = (S^1 \times \mathbb{C}, \|.\|, \nabla^\lambda)$ of holonomy $\exp(2\pi i \lambda)$, $\lambda \in \mathbb{R}$. Then $\not{D}_{S^1} \otimes \mathbf{V} = i\partial_t - 2\pi i(\frac{1}{2} + \lambda)$. The summand $\frac{1}{2}$ comes from the spin structure which is non-bounding. In this case we get

$$\rho(S^1, \nabla^\lambda, s) = [-\lambda] .$$

46.1.4 An index theorem

Let $b \in \pi_2(\mathbf{KU})$ be the Bott element.

Proposition 46.2. *If $n = 2m - 1$, then we have*

$$\rho(B, \nabla, s) b^m = \mathtt{reg}_{\mathbb{C}}(c(B, \nabla, s)) .$$

In the example above we have $c(S^1, \nabla^\lambda, s) \cong \exp(2\pi i \lambda) \in \mathbb{C}^*$ and this is mapped to $[-\lambda]b$ under $\mathtt{reg}_{\mathbb{C},1}$.

46.2 Foliations

46.2.1 Generalization of the analytic side

Assume that B is a closed manifold and has a foliation $\mathcal{F} \subseteq TB$ of dimension f, i.e a subbundle such that $\Gamma(B, \mathcal{F})$ is closed under commutators. We assume that $\mathcal{F}$ has a stable framing $s : \mathcal{F} \oplus \underline{\mathbb{R}}^k \cong \underline{\mathbb{R}^{k+n}}$. We further consider a complex vector bundle $V \to B$ with a partial flat connection ∇^I in the $\mathcal{F}$-direction. The case where $\mathcal{F} = TB$ is considered above. In the general situation we choose an extension ∇ of ∇^I to a connection.

The normal bundle $\mathcal{F}^\perp := TB/\mathcal{F}$ has a canonical flat partial connection $\nabla^{\mathcal{F}^\perp, I}$ in the $\mathcal{F}$-direction. The Riemannian metric on B induces a decomposition $\mathcal{F} \oplus \mathcal{F}^\perp \cong TB$. We choose any extension $\nabla^{\mathcal{F}^\perp}$ of $\nabla^{\mathcal{F}^\perp, I}$. We extend definition of the ρ-invariant as follows.

Definition 46.3.

$$\rho(B, \nabla^I, \mathcal{F}, s) := \xi(\not{D}_B \otimes \mathbf{V}) + \left[\int \hat{\mathbf{A}}(\nabla^{LC}) \wedge \widetilde{\mathbf{ch}}(\nabla, \nabla^u) - \tilde{\hat{\mathbf{A}}}(\nabla^{LC}, \nabla^s \oplus \nabla^{\mathcal{F}^\perp}) \wedge \mathbf{ch}(\nabla)\right]_{\mathbb{C}/\mathbb{Z}} .$$

Proposition 46.4. *If $2f - 1 \geq n$, then $\rho(B, \nabla^I, \mathcal{F}, s) \in \mathbb{C}/\mathbb{Z}$ does not depend on the additional choices. It is a bordism invariant of $(B, \nabla^I, \mathcal{F}, s)$.*

Our goal is to understand this spectral-geometric invariant topologically.

46.2.2 The higher regulator

Let X be a closed manifold and $\mathbf{K}(C^\infty(X))$ be the algebraic K-theory spectrum of the algebra $C^\infty(X)$.

Proposition 46.5. *For $f \geq \dim(X) + 1$ there exists a regulator map*

$$\mathtt{reg}_{C^\infty(X)} : \pi_f(\mathbf{K}(C^\infty(X))) \to \mathbf{KU}\mathbb{C}/\mathbb{Z}^{-f-1}(X) .$$

We now assume that X is spin. Then X is oriented for $\mathbf{KU}$ and we have an integration

$$\int_X : \mathbf{KU}\mathbb{C}/\mathbb{Z}^{-f-1}(X) \to \pi_{f+1+\dim(X)}(\mathbf{KU}\mathbb{C}/\mathbb{Z}) .$$

46.2.3 The conjecture

We assume that $B = M \times X$, M is stably framed, $\dim(M) = f$ and $\mathcal{F} = TM \boxplus \{0\}$. The partial flat connection ∇^I induces a map of spaces

$$v : M \to BGL(\ell, C^\infty(X)^\delta) \to BGL(C^\infty(X)^\delta)^+ \cong \Omega^\infty \mathbf{K}(C^\infty(X)) .$$

We get a class

$$[M, s, v] \in \pi_f(\mathbf{S} \wedge \Omega^\infty \mathbf{K}(C^\infty(X)))$$

and its stabilization

$$c(M, \nabla^I, s) \in \pi_f(\mathbf{K}(C^\infty(X))) .$$

We finally get

$$\int_X \mathtt{reg}_{C^\infty(X)}(c(M,\nabla^I),s) \in \pi_{n+1}(\mathbf{KU}\mathbb{C}/\mathbb{Z}) \ ,$$

where $n = \dim(B) = f + \dim(X)$.
Let us assume that the spin structure on $M \times X$ is induced from that of X and the framing on M.

Conjecture 46.6. *If $m = 2n-1$ and $2f \geq n+1$, then we have the equality*

$$\int_X \mathtt{reg}_{C^\infty(X)}(c(M,\nabla^I,s)) = b^m \rho(M \times X, \nabla^I, TM \boxplus \{0\}, s) \ .$$

46.3 More conjectures

46.3.1 The Connes-Karounbi character

We assume that X is a closed spin manifold of dimension d. The Dirac operator on X twisted by the geomtric bundle $\mathbf{V}$ induces a $d+1$-summable Fredholm module

$$(L^2(X, S(X) \otimes V), F)$$

over $C^\infty(X)$, where $F = (\not{D}_X \otimes \mathbf{V})((\not{D}_X \otimes \mathbf{V})^2 + 1)^{-1/2}$.

Let $\mathcal{M}_d$ be the classifying algebra for $d+1$-summable Fredholm modules. It is given by

$$\mathcal{M}_d = \{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mid a_{11}, a_{22} \in B(\ell^2) \ , \ a_{12}, a_{21} \in \mathcal{L}^{d+1}(\ell^2) \} \ .$$

There is a map (fixed by the choices of identifications $\mathtt{im}(P^\pm) \cong \ell^2$ which are unique up to unitary isomorphism)

$$b_{\not{D}_X} : C^\infty(X) \to \mathcal{M}_d \ , \quad b_{\not{D}_X}(f) := \begin{pmatrix} P^+ f P^+ & P^+ f P^- \\ P^- f P^+ & P^- f P^- \end{pmatrix} \ .$$

This homomorphism of algebras induces a map of spectra

$$b_{\not{D}_X} : \mathbf{K}(C^\infty(X)) \to \mathbf{K}(\mathcal{M}_d) \ .$$

The Connes-Karoubi character is a homomorphism

$$\delta : \pi_{d+1}(\mathbf{K}(\mathcal{M}_d)) \to \mathbb{C}/\mathbb{Z} \ .$$

We therefore get a homomorphism

$$\delta \circ b_{\not{D}_X} : \pi_{d+1}(\mathbf{K}(C^\infty(X))) \to \mathbb{C}/\mathbb{Z} \ .$$

Conjecture 46.7. *We have the equality*

$$\delta \circ b_{\not{D}_X} = \int_X \circ \mathtt{reg}_{C^\infty(X),d} : \pi_{d+1}(\mathbf{K}(C^\infty(X))) \to \mathbb{C}/\mathbb{Z} \ .$$

Remark 46.8. I think that I can show the equality

$$\delta \circ b_{\not{D}_X}(c(M, \nabla, s)) = \rho(M \times X, \nabla^I, TM \boxplus \{0\}, s) \ .$$

So Conjecture 46.7 would imply Conjecture 46.6. □

46.3.2 Relative K-theory

We consider the functor $X \mapsto \mathbf{K}(C^\infty(X))$ from manifolds to spectra. There is a natural way to construct a homotopy invariant version $\mathbf{K}^{top}(C^\infty(X))$. We have a fibre sequence

$$\Sigma^{-1}\mathbf{K}^{top}(C^\infty(X)) \to \mathbf{K}^{rel}(C^\infty(X)) \xrightarrow{\partial} \mathbf{K}(C^\infty(X)) \to \mathbf{K}^{top}(C^\infty(X))$$

which defines the relative algebraic K-theory of $C^\infty(X)$.

Theorem 46.9. *We have the equality*

$$\delta \circ b_{\not{D}_X} \circ \partial = \int_X \circ \mathtt{reg}_{C^\infty(X),d} \circ \partial : \pi_{d+1}(\mathbf{K}^{rel}(C^\infty(X))) \to \mathbb{C}/\mathbb{Z} \ .$$

Remark 46.10. I do not have any example of a class in $\pi_{d+1}(\mathbf{K}(C^\infty(X))$ which is not in the image of ∂.

46.4 Construction of the regulator

46.4.1 Chern characters

We start with the Goodwillie-Jones Chern character

$$\mathbf{K}(C^\infty(X)) \to \mathbf{CC}^-(C^\infty(X))$$

from algebraic K-theory to negative cyclic homology. We define

$$\mathbf{DD}^-(X) := H(\prod_{p\in\mathbb{Z}} \sigma^{\geq p}\Omega(X)[2p])$$

and consider the comparison map

$$\mathbf{CC}^-(C^\infty(X)) \to \mathbf{DD}^-(X)$$

from negative cyclic homology of a smooth manifold with differential forms. We get a diagram

$$\begin{array}{ccc} \mathbf{K}(C^\infty(X)) & \longrightarrow & \mathbf{DD}^-(X) \\ \big\downarrow & & \big\downarrow \\ \mathbf{ku}(X) & \xrightarrow{\mathbf{ch}_{|\mathbf{ku}}} & \mathbf{DD}(X) \end{array} \ . \tag{23}$$

Here $\mathbf{DD}(X) := H(\prod_{p\in\mathbb{Z}} \Omega(X)[2p])$ represents the 2-periodic cohomology of X. The lower map is obtained by forcing homotopy invariance and descent on both sides of the upper map. It is equivalent to the usual Chern character.

46.4.2 Differential K-theory

We define differential $\mathbf{KU}$-theory of X by the pull-back

$$\begin{array}{ccc} \widehat{\mathbf{KU}}(X) & \xrightarrow{R} & \mathbf{DD}^-(X) \\ \big\downarrow & & \big\downarrow \\ \mathbf{KU}(X) & \xrightarrow{\mathbf{ch}} & \mathbf{DD}(X) \end{array} \ .$$

Definition 46.11. *Using* (23) *we define a regulator map*

$$\mathtt{reg}_{C^\infty(X)} : \mathbf{K}(C^\infty(X)) \to \widehat{\mathbf{KU}}(X) \ .$$

The regulator can actually be refined to a map of commutative ring spectra. The map (22) is the special case $X = *$ (and restriction to the connective covering).

46.4.3 High degrees

The following two Lemmas are shown by calculation.

Lemma 46.12. *If $f \geq \dim(X)$, then we have a natural isomorphism*

$$\mathbf{KU}\mathbb{C}/\mathbb{Z}^{-f-1}(X) \cong \mathtt{ker}\Big(R : \pi_f(\widehat{\mathbf{KU}}(X)) \to \ \pi_f(\mathbf{DD}^-(X))\Big) \ .$$

Lemma 46.13. *If $f \geq \dim(X) + 1$, then*

$$\mathtt{reg}_{C^\infty(X),k} : \pi_f(\mathbf{K}(C^\infty(X)) \to \pi_f(\widehat{\mathbf{KU}}(X))$$

has values in $\mathtt{ker}(R)$*.*

Hence for $f \geq \dim(X) + 1$ we get the regulator map asserted in Proposition 46.5.

$$\mathtt{reg}_{C^\infty(X),f} : \pi_f(\mathbf{K}(C^\infty(X)) \to \mathbf{KU}\mathbb{C}/\mathbb{Z}^{-f-1}(X) \ .$$