Homotopietheorie Seminar (S2D4)
Topological K-theory

Mondays, 12:15-13:45 in N0.007
Organisational meeting on Tuesday, July 15, 2025, at 10:15 in N0.007.

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Topological K-theory, just called K-theory from now on, is a cohomology theory built from vector bundles. The goal for this seminar is to study vector bundles, define the cohomology of K-theory, and then prove statements in algebra and topology using this theory. The main references are the books of Atiyah [Ati66b], Hatcher [Hat], and Karoubi [Kar78]. We will mostly focus on complex K-theory. [1]

Each talk should be roughly 80 minutes long, accounting for questions and comments, and so it is up to each presenter to choose exactly what should be presented from topic, although the main theorems and definitions should always be given. Make sure to include a good amount of examples in your talk.

(20.10.2025, Florian Jänicke) Vector bundles I Define vector bundles, both complex and real, give some examples, make the basic constructions including the pullback, direct sum, tensor product, exterior algebra, etc. Prove that over a compact Hausdorff space, all vector bundles embed into a trivial bundle. See [Ati66b, §1.1-1.2], [Hat, §1.1], and [Kar78, §I].

(27.10.2025, Leo Bergmann) Vector bundles II Prove the homotopy invariance of vector bundles and their classification over compact Hausdorff spaces in terms of homotopy classes of maps into a Grassmannian. Discuss clutching functions and the classification of vector bundles over spheres. See [Ati66b, §1.3], [Hat, §1.2], and [Kar78, §I].

(03.11.2025, Bendix Pawig) The ring $K(X)$ Define the K-theory group $K(X)$ for a compact Hausdorff space X and show it is a ring. Define the reduced group $\widetilde{K}(X)$. Prove that $K(-)$ satisfies the axioms of a cohomology theory defined only in degree zero. See [Ati66b, §2.1 & 2.4] and [Hat, §2.1-2.2], and [Kar78, §II].

(10.11.2025, Jan Malmström) The fundamental product theorem State and prove the theorem that the exterior product map $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X. See [Ati66b, §2.2], [Hat, §2.1], and [Kar78, §III].

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[1] This syllabus is based on a previous seminar organized by Jack Davies and Liz Tatum. We thank them for letting us use it.

(17.11.2025, Linus Wessing) The cohomology theory $K^*(X)$ Use the fundamental product theory to define the groups $K^n(X)$ for all $n \in \mathbf{Z}$, prove Bott periodicity, and show the resulting functor $K^*(-)$ defines a cohomology theory. Show that $K^n(X)$ is represented by a sequence of spaces K_n, meaning that there are isomorphisms $K^n(X) \cong [X, K_n]$ for compact Hausdorff spaces X, together with homotopy equivalences $\Omega K_{n+1} \simeq K_n$. See [Ati66b, §2.2 & 2.4], [Hat, §2.2], and [Kar78, §III].

(24.11.2025, Felix Gervasi) The splitting principle and the Leray–Hisch theorem Prove the splitting principle and the Leray–Hirsch theorem for K-theory. Use this to compute the complex K-theory of complex projective spaces and Grassmannians. See [Hat, §2.3] and [Kar78, §IV.2].

(01.12.2025, Tong Zhang) Adams operations and λ-ring structures Define λ-rings, construct the Adams operations on $K(X)$, and combine these results to show that $K(X)$ has a natural structure of a λ-ring. Calculate the λ-ring structure on spheres and projective spaces. See [Ati66b, §3.1-3.2], [Hat, §2.3], and [Yau10].

(08.12.2025, Tobias Schubert) Hopf invariant one theorem Show that the Hopf invariant one theorem implies the nonexistence of H-space structure of spheres, and hence the parallelisability of spheres and division algebra structure of Euclidean space. Prove the Hopf invariant one theorem using Adams operations and complex K-theory. See [Hat, §2.3] and [Kar78, §V.1].

(15.12.2025, Benjamin Legerer) K-theory and reality Following Atiyah [Ati66a], construct a C_2-equivariant cohomology theory KR, prove equivariant Bott periodicity, and use this to conclude Bott periodicity for real K-theory. See [Ati66a].

(12.01.2026, Zehao Yang) Clifford algebras Describe the relationship between Clifford algebras, Fredholm operators, and topological K-theory, in particular with regard to real topological K-theory. Discuss how these can be used to give a reproof of Bott periodicity. See [ABS64] and [Kar78, §II.1 & III.3].

(19.1.2026, Joachim Treczoks) Chern classes and the Chern character Define the Chern classes $c_n(E) \in H^{2n}(B; \mathbf{Z})$ of a complex vector bundle E over B and show their defining properties. Construct the Chern character $K(X) \to H^*(X; \mathbf{Q})$ and show that it is a rational isomorphism on finite CW-complexes. See [Hat, §3.1 & 4.1].

(26.1.2026, Juri Kaganskiy) Hirzebruch–Riemann–Roch theorem State and prove the Hirzebruch–Riemann–Roch theorem. See [Kar78, §V.4].

(2.2.2026, Fynn Herbermann) Vector fields on spheres Discuss the result of Adams [Ada62] which calculates the precise maximal number of linearly independent trangent vector fields on spheres. See [Kar78, §V.2] for an overview.

References

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl):3–38, 1964.

[Ada62] J. F. Adams. Vector fields on spheres. *Ann. of Math. (2)*, 75:603–632, 1962.

[Ati66a] M. F. Atiyah. K-theory and reality. *Quart. J. Math. Oxford Ser. (2)*, 17:367–386, 1966.

[Ati66b] M. F. Atiyah. Power operations in K-theory. *Quart. J. Math. Oxford Ser. (2)*, 17:165–193, 1966.

[Hat] A. Hatcher. *Vector bundles and K-theory.* Available at https://pi.math.cornell.edu/ hatcher/VBKT/VBpage.html.

[Kar78] Max Karoubi. *K-theory*, volume Band 226 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin-New York, 1978. An introduction.

[Yau10] Donald Yau. *Lambda-rings.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.